# A boundary value problem in linear spaces

Y. Tagamlitzki

A compact, convex subset $K$ of a locally convex linear space is called a topological simplex if every continuous real-valued function on the set $E(K)$ of the extreme points of $K$ admits a continuous, affine continuation in $K$.

It is well-known that $K$ is a topological simplex if and only if it is a simplex in the sense of Choquet and the set $E(K)$ of its extreme points is closed (and therefore compact).

We start by stating the following theorem, which we shall use latter.

Theorem 1. If $K$ is a topological simplex, then every continuous map of the set $E(K)$ of the extreme points of $K$ into a convex, compact subset of a locally convex linear space admits a continuous, affine continuation in $K$.

<u>Proof.</u> Let $\varphi(p)$ be a continuous mapping of $E(K)$ into a compact, convex subset $N$ of a linear locally convex space $Y$ and let $g(y)$ be a continuous linear real-valued functional in $Y$. Then $g(\varphi(p))$ is a continuous real-valued function on $E(K)$. Let $\Theta_g(x)$ be its continuous, affine continuation in $K$ and let $L$ be the set of the points $X$ of $K$, for which the equation

$$\Theta_g(x) = g(y)$$

admits a solution $y$ in $N$ for every $g$, which does depend on $g$.

Since $E(K) \subset L$ holds, we find $K \subset L$. On the other hand $N$ is compact and therefore $y$ is a continuous function of $x$. This function is the desired continuation of $\varphi(p)$.

<u>Definition.</u> Let $m$ be an affine, continuous mapping from $K$ into $K$ and let $\ell$ be a real-valued, affine function $K$. The couple $(m, \ell)$ is said to ba a decomposition pair if in $K$

$$0 \leqq \ell(x) \leqq 1$$

$$m(x) \in \ell(x) K$$

$$x - m(x) \in (1-\ell(x)) K$$

holds. We sometimes call $(m, \ell)$ a decomposing pair.

It is said that the points $p$ and $q$ are separated by a decomposition pair $(m, \varphi)$ if $\varphi(p) \neq \varphi(q)$ holds.

Theorem 2. A compact, convex set $K$ in a locally convex linear space is a topological simplex if and only if the decomposing pairs separate the extreme points of $K$.

For the proof we need several lemmas.

Lemma 1. Let $K$ be convex and compact and $(m, \varphi)$ be a decomposing pair in $K$. If $S$ is an extreme point of $K$ and $\mu$ is a real number such that $0 \leqq \mu \leqq 1$ holds, then the set of the points $x$ of $K$ satisfying the equation

$$m(x) = \mu\, x + (\varphi(x) - \mu)\, S$$

is an extreme subset of $K$.

Proof. We put

$$\ell(x) = m(x) - \varphi(x)\, S$$

and

$$n(x) = \ell(\ell(x) + S) - 2\mu\, \ell(x) + \mu^2\, (x - S) + S.$$

If $p$ is an extreme point of $K$, then

$$m(p) = \varphi(p)\, P$$

holds. Thus $\ell(p) = \varphi(p)\,(p-s)$.

Since $n(x) \in K$ holds for the extreme points of $K$, it holds everywhere in $K$. Since $s$ is an extreme point of $K$ the set $A$ of the points $x$ of $K$ satisfying $n(x) = s$ is an extreme subset of $K$. On the other hand this set coincides with the set $B$ of the points $x$ satisfying

$$m(x) = \mu + (\quad (x) - u)\, s.$$

Indeed, since the implication $A \supset B$ is trivial we need only consider the implication $B \supset A$. Now observe that the extreme points of $A$ belong to $B$ and this completes the proof since $A$ is an extreme subset of $K$.

Lemma 2. If the decomposing pairs $(m, \varphi)$ separat the extreme elements of the convex, compact set $K$, the set $E(k)$ of the extreme points of $K$ is closed (and therefore compact).

Proof. Let $\{p_\alpha\}$ be a directed system of extreme points of $K$ converging to $p_0$. We denote by $F_{m,\varphi,s}$ the set of the points $x$ of $K$ satisfying the equality

$$m(x) - \varphi(x)\, S = \varphi(po)\ (x-s).$$

By lemma 1 it is an extreme subset of K. Clearly

$$po \in F_{m,\varphi,s}$$

since $m(po) = \varphi(po)\, po$. In order to prove that po is an extreme point of K we shall show that the intersection $F$ of the extreme sets $F_{m,\varphi,s}$ has one single extreme point, which therefore coincides with po. This extreme point of $F$ is an extreme point of K as well, since $F$ is an extreme subset of K. Suppose there are two different extreme points $p_1$ and $p_2$. Since the decomposing pairs separate the extreme points, there is a decomposing pair $(m, \varphi)$ such that $\varphi(p_1) \neq (p_2)$ holds. We may suppose that $\varphi(p_o) \neq (p_o)$. Clearly $F$ is an extreme subset of K. Hence $p_1$ is an extreme point of K and therefore $m(p_1) = \varphi(p_1)\, p_1$. On the other hand

$$m(p_1) - \varphi(p_1)\, s = \varphi(po)\ (p_1 - s),$$

Hence

$$\varphi(p_1)\ (p_1 - s) = \varphi(po)\ (p_1 - S),$$

which is a contradiction since we may choose $S = p_2$.

<u>Lemma 3.</u> Let $(m_1\ \varphi)$ be two decomposing pairs. Then there is such a decomposing pair $(m,\varphi)$ that for the extreme points p of K.

$$\varphi(p) = \varphi_1(p)\ \varphi_2(p)$$

holds.

<u>Proof.</u> Let S be an extreme point of K. Then the couple of the functions.

We write

$$f_1(x) = m_1(x) - \varphi_1(x)\, S$$
$$f_2(x) = m_2(x) - \varphi_2(x)\, S.$$

Then the couple of the functions.

$$m(x) = f_2(f_1(x) + S) + \varphi(x)\, S$$

$$\varphi(x) = {}_2\varphi(f_1(x) + S) - (1 - \varphi_1(x)\varphi_2(S)$$

is a decomposing pair with the desired property since for the extreme points p of K.

$$f_1(p) = \varphi_1(p)\ (p-s) \qquad f_2(p) = \varphi_2(p)\ (p-s)$$

$$\varphi(p) = \varphi_1(p)\ \varphi_2(p)$$

$$m(p) = \varphi_1(p)\, \varphi_2(p)\, P$$

Holds.

Proof of theorem 2. Let K be a convex, compact subset of a locally compact linear space. Suppose that the decomposing pairs separate the extreme points of K. We shall prove that every continuous real-valued function f(p) on the set E(K) of the extreme points of K admits a continuous, affine continuation on K. By lemma 2 the set E(K) is compact. Now consider the linear hull R of the real-valued affine, continuous functions $\varphi(x)$ on K such that m (x) can be chosen so as the couple $(m, \varphi)$ to be a decomposing pair. By lemma 3 the linear space R is an algebra on the set E (K) of the extreme points of K. Hence by the Stone-Weierstrass theorem we can find in R a function $f_n$ satisfying

$$|f(p) - f_n(p)| < \frac{1}{n}$$

on E (K) . Hence

$$f_m(p) - f_n(p)| < \frac{1}{n} + \frac{1}{m} ,$$

On the other hand the functions of R are continuious and affine and therefore

$$| f_n(x) - f_m(x)| < \frac{1}{n} + \frac{1}{m}$$

holds in K as well. Clearly $\lim f_n(x)$ is the desired continuation of f (p).

Now we shall prove the converse. Suppose that every continuious function on the set of the extreme points of K admits a continuuous affine continuation on K. Let $p_1$ and $p_2$ be two different extreme points. Let f(x) be a continuous linear functional which separates $p_1$ and $p_2$ .

Since f(x) is bounded on K we may choose the constants $\alpha$ and B so as to have $0 < \alpha f(x) + \beta < 1$ on K. On the other hand the function

$$(\alpha f(p) + B) P$$

is a continuous map of the set E (K) of the extreme points of K into the compact and convex set $U \lambda K$.

$$0 \leqq \lambda \leqq 1$$

Hence by theorem 1 this map admits a continuous, affine continuation m (x) on K. Then the couple $(m(x), \alpha f(x) + B)$ is a decomposing pair which separates the points $p_1$ and $p_2$ .