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THE PRINCIPLE OF TOPOLOGICAL INDUCTION*

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Ярослав Тагамлицкий. **Принцип топологической индукции.** В работе излагается одно существенное обобщение известной теоремы Крейна — Мильмана из функционального анализа. Это обобщение излагалось в 1974/1975 году в лекциях покойного профессора Я. Тагамлицкого, прочитанных в Софийском университете, и оно является модифицированной версией одного более раннего обобщения, сделанного тем же автором. Вместо компактного подмножества локально выпуклого топологического линейного пространства рассматривается произвольное компактное топологическое пространство, оснащенное некоторым семейством открытых множеств, удовлетворяющим определенным условиям. В этой общей ситуации определяется подходящий квази-порядок на рассматриваемом компактном пространстве. Доказывается, что каждый раз, когда некоторое множество из упомянутого семейства открытых множеств содержит все минимальные элементы пространства в смысле Цорна, это множество содержит все точки пространства. Теорема Крейна — Мильмана может быть получена как частный случай путем рассмотрения семейства, множества которого — все пересечения открытых выпуклых подмножеств данного топологического линейного пространства с данным компактным подмножеством этого пространства.

Yaroslav Tagamlitzki. **The principle of topological induction.** The paper presents an essential generalization of the well-known Krein-Milman theorem of Functional Analysis. This generalization has been exposed in the lectures, delivered by the late Professor Y. Tagamlitzki at the Sofia University in 1974/1975, and it is a modified version of an earlier generalization due to the same author. Instead of a compact subset of a locally convex topological linear space, an arbitrary compact topological space is considered which is supplied with a family of open subsets satisfying certain conditions. In this general situation, a suitable quasi-order is defined in the compact space under consideration. It is proved that whenever a set from the introduced family of open subsets contains all minimal elements of the space in the sense of Zorn, then this set contains the whole space. The Krein-Milman theorem may be obtained as a particular case by considering a family, whose sets are the intersections of the open convex subsets of the given topological linear space with the given compact subset of the space.

In [2] a generalization of the famous Krein-Milman theorem for the case of arbitrary compact topological space is presented and some applications of this generalization are pointed out. The generalization has been named "topological induction". In this paper a modified version of the topological induction will be exposed, which is in accordance with the lectures [1].

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Let us first recall some well-known definitions. As usually, a quasi-order on a set is a reflexive and transitive binary relation defined in it.

Definition 1. If $\leqq$ is a quasi-order in a set Z, an element p in Z will be called minimal in the sense of Zorn (with respect to $\leqq$) if and only if all z in Z, satisfying $z\leqq p$, satisfy also $p\leqq z$. If we exchange $z\leqq p$ and $p\leqq z$ in the last condition, then we get the notion of a maximal element in the sense of Zorn.

A multivalued function from a set X into a set Y is, by definition, a function from X into the set of subsets of Y. For such a function f, the multivalued function f^{-1} from Y into X is defined by

$$f^{-1}(y)=\{x\in X;\quad y\in f(x)\}.$$

From now on, we suppose that two topological spaces X and Y are given, as well as a quasi-order $\leqq$ in X and a multivalued function f from X into Y, which satisfy the following conditions:

(i) each monotonically increasing transfinite sequence of elements of X is convergent to some element of X (for the notion of convergence of generalized sequences, cf. e. g. [3], Ch. 2]);

(ii) The set $\{x\in X;\ a\leqq x\}$ is closed for every $a\in X$;

(iii) f is monotonically increasing, i. e. whenever $x_1, x_2\in X$ and $x_1\leqq x_2$, we have $f(x_1)\subset f(x_2)$;

(iv) the subset $f(x)$ of Y is open for every $x\in X$;

(v) the subset $f^{-1}(y)$ of X is open for every $y\in Y$.

The reader, who is interested in the history of the topological induction, should notice that the spaces X, Y and the function f correspond to the set Γ, the space M and the function φ from [2], respectively. The connection of the above assumptions with those in [2] becomes clearer by the following lemma, which will be also necessary for our further considerations (the proof will be given after the formulation of the main result):

Lemma 1. Let $\{x_g\}_{g\in G}$ (where G is a segment of ordinals) be a monotonically increasing transfinite sequence of elements of X. Then there is an element a of X such that $x_g\leqq a$ for all g in G and the inclusion

$$f(a)\subset\bigcup\{f(x_g);\ g\in G\}$$

holds. □

Remark 1. As a matter of fact, the equality $f(a)=\bigcup\{f(x_g);\ g\in G\}$ will hold for each element a possessing the properties formulated in Lemma 1. Indeed, $x_g\leqq a$ and condition (iii) imply that $f(x_g)\subset f(a)$. Therefore,

$$\bigcup\{f(x_g);\ g\in G\}\subset f(a).$$

Remark 2. Instead of assuming a topology on X and conditions (i), (ii) and (v) satisfied, we could simply assume the statement of Lemma 1 (compare with [2]). This would be an approach equivalent to the above one, since the statement of Lemma 1 and the monotonicity of f imply that (i) is satisfied for the weakest topology on X, which satisfies (ii) and (v).

Definition 2. For every subset S of Y we define the set $(S)\subset Y$ in the following way: an element y in Y belongs to (S) if and only if

(a) for every $x\in X$, the condition $f(x)\supset S$ implies $y\in f(x)$;

(b) whenever $x_1\in X$, $f(x_1)\cap S\neq\emptyset$ and $S\setminus f(x_1)\neq\emptyset$, then there is an $x_2\in X$ such that $x_2\geqq x_1$, $y\in f(x_2)$ and $S\setminus f(x_2)\neq\emptyset$. □

Definition 3. In the space Y we define a relation $\leqq$ in the following way: $y_1 \leqq y_2$ if and only if there is a set $S \subset Y$ such that $y_1 \in S$ and $y_2 \in (S)$.□

The main result of this paper is the following

Theorem (Principle of Topological Induction). The relation $\leqq$ in Y is a quasi-order in Y. If in addition Y is compact, then its set Ex of minimal elements in the sense of Zorn is not empty and has the following properties:

1. If $p \in \mathrm{Ex}$ then $y \leqq p$ implies $y \approx p$, i. e. for every $x \in X$, $y \in f(x)$ if and only if $p \in f(x)$.

2. The inclusion $f(x) \supset \mathrm{Ex}$ implies $f(x) \supset Y$.□

The proof of this theorem decomposes in several lemmas.

The first one is Lemma 1 which was already formulated.

Proof of Lemma 1. By (i), the sequence $\{x_g\}_{g \in G}$ converges of some element a of X. If $g_0 \in G$, then the set $\{x \in X; x_{g_0} \leqq x\}$ is closed by (ii) and contains x_g for $g \geqq g_0$; hence a belongs to this set, i.e. $x_{g_0} \leqq a$. Consider now an arbitrary element y of $f(a)$. Then $a \in f^{-1}(y)$, which is an open set by (v). Therefore, $x_g \in f^{-1}(y)$ for sufficiently large $g \in G$. Thus $y \in f(x_g)$ for some $g \in G$, i.e. $y \in \cup \{f(x_g), g \in G\}$. Hence, $f(a) \subset \cup \{f(x_g);\ g \in G\}$.□

Lemma 2. The relation $\leqq$ in Y, introduced by Definition 3, is a quasi-order on Y.□

Proof. For every $y \in Y$ it is easy to check that $y \in (\{y\})$. Consequently, the relation is reflexive.

For proving transitivity, let for some y_1, y_2, y_3 in Y we have $y_1 \leqq y_2$ and $y_2 \leqq y_3$. There exist sets $S_1, S_2 \subset Y$ such that $y_1 \in S_1$, $y_2 \in (S_1)$ and $y_2 \in S_2$, $y_3 \in (S_2)$.

Consider the set $S = S_1 \cup S_2$. We shall prove that $y_3 \in (S)$, which is sufficient for proving $y_1 \leqq y_3$.

a) If $f(x_1) \supset S$ then $f(x_1) \supset S_2$, hence $y_3 \in f(x_1)$.

b) Let for some $x_1 \in X$ we have $f(x_1) \cap S \neq \varnothing$ and $S \setminus f(x_1) \neq \varnothing$.

There are several possibilities:

1. If $f(x_1) \supset S_2$ then $y_3 \in f(x_1)$, and we set $x_2 = x_1$.

2. Let $f(x_1) \cap S_2 \neq \varnothing$ and $S_2 \setminus f(x_1) \neq \varnothing$. Since $y_3 \in (S_2)$, there exists $x_2 \geqq x_1$ such that $y_3 \in f(x_2)$ and $S_2 \setminus f(x_2) \neq \varnothing$. The last implies $S \setminus f(x_2) \neq \varnothing$.

3. Let $f(x_1) \cap S_2 = \varnothing$. Then $f(x_1) \cap S_1 \neq \varnothing$.

Now there are two possibilities for the last case too.

3.1. Let $f(x_1) \supset S_1$. Then $y_2 \in f(x_1)$. But we have also $y_2 \in S_2$. This is incompatible with 3.

3.2. Let $f(x_1)$ does not contain S_1. Since $y_2 \in (S_1)$, there is some $x_3 \geqq x_1$ for which $y_2 \in f(x_3)$ and $S_1 \setminus f(x_3) \neq \varnothing$. The last $S \setminus f(x_3) \neq \varnothing$, and $y_2 \in f(x_3)$ implies $S_2 \setminus f(x_3) \neq \varnothing$, and hence there is an $x_2 \geqq x_3$ such that $y_3 \in f(x_2)$ and $S_2 \setminus f(x_2) \neq \varnothing$. Obviously, $x_2 \geqq x_1$ and $S \setminus f(x_2) \neq \varnothing$.□

Lemma 3. The st $\{y \in Y;\ y \leqq b\}$ is closed for every $b \in Y$.□

*Proof.** Let $b \in Y$ and $B = \{y \in Y;\ y \leqq b\}$. By the definition of the relation $\leqq$ in Y, the set B is the union of all subsets S of Y such that $b \in (S)$. We are aiming to prove that $b \in ([B])$, where $[B]$ is the closure of B. This will be sufficient for the proof of the lemma, since $b \in ([B])$ will imply $[B] \subset B$ (according to the above characterization of B). The element b and the set $[B]$ obviously satisfy the first condition of Definition 2: if $x \in X$ and $[B] \subset f(x)$ then $b \in f(x)$, since $b \in B \subset B[B]$. For proving that the second condition is also satisfied, suppose $x_1 \in X$, $[B] \cap f(x_1) \neq \varnothing$ and $[B] \setminus f(x_1) \neq \varnothing$. Since $f(x_1)$ is an open set, $[B] \cap f(x_1) \neq \varnothing$ implies that

* This proof is given by Dimiter Skordev and it differs from the original proof in [1] which uses the Axiom of Choice.

$B\cap f(x_1)\neq\emptyset$. Using the given characterization of B once more, we conclude that $S\cap f(x_1)\neq\emptyset$ for some $S\subset B$ such that $b\in(S)$. If $S\setminus f(x_1)\neq\emptyset$, then there is an $x_2\in X$ such that $x_1\leqq x_2$, $b\in f(x_2)$ and $S\setminus f(x_2)\neq\emptyset$. But $S\setminus f(x_2)\neq\emptyset$ evidently implies $[B]\setminus f(x_2)\neq\emptyset$ So, in this case, we proved the existence of an element $x_2\in X$ such that $x_1\leqq x_2$, $b\in f(x_2)$ and $[B]\setminus f(x_2)\neq\emptyset$. Such an x_2, namely $x_2=x_1$, exists also when $S\setminus f(x_1)=\emptyset$, since in this case $b\in f(x_1)$ due to the inclusion $S\subset f(x_1)$. The proof is completed.□

Let us note that the previous lemmas were proved for an arbitrary topological space Y. From now on, it becomes important that Y is a compact topological space, and we shall assume this.

Lemma 4. For each b in Y, there is an element $p\in Y$ such that $p\leqq b$ and p is minimal in the sense of Zorn.□

Proof. Since the rays $\{y\in Y;\ y\leqq b\}$ are closed by Lemma 3, and Y is compact by assumption, the proof easily follows through the Lemma of Zorn.□

Lemma 5. Let $a_0\in X$, B be a closed subset of Y and $B\setminus f(a_0)\neq\emptyset$. Then there is some a_1 in X such that $a_1\geqq a_0$ and a_1 is a maximal element in the sense of Zorn in the set $\{x\in X;\ B\setminus f(x)\neq\emptyset\}$.□

Proof. According to the Lemma of Zorn, it is sufficient to prove that each monotonically increasing transfinite sequence of elements of the set $\{x\in X;\ B\setminus f(x)\neq\emptyset\}$ has an upper bound, belonging to this set. Suppose $\{x_g\}_{g\in G}$ is such a transfinite sequence. By Lemma 1, there is an upper bound a of $\{x_g\}_{g\in G}$ in X such that

$$f(a)\subset\bigcup\{f(x_g);\ g\in G\}$$

We shall prove that $B\setminus f(a)\neq\emptyset$. Suppose the contrary: $B\subset f(a)$. Then $B\subset\cup\{f(x_g);\ g\in G\}$. Since B is compact, there is a finite number of elements $g_1,g_2,\ldots,g_n$ of G such that

$$B\subset\bigcup_{i=1}^{n}f(x_g).$$

This results in a unique index g_0 (namely, the greatest among $g_1,g_2,\ldots,g_n$) such that $B\subset f(x_{g_0})$, and this is impossible since $x_{g_0}\in\{x\in X;\ B\setminus f(x)\neq\emptyset\}$.□

The notion of a left Dedekind class, which will be introduced below, plays the role of the notion of extreme subset, the latter being widely used in other approaches to problems of convexity and extreme points.

Definition 4. A subset B of Y is called a left Dedekind class if and only if $\{y\in Y;\ y\leqq b\}\subset B$ for every b in B.

The simplest examples of left Dedekind classes in Y are the set Y itself and all rays $\{y\in Y;\ y\leqq b\}$, where $b\in Y$.

Lemma 6. Let $a_0\in X$, B be a closed left Dedekind class in Y and $B\setminus f(a_0)\neq\emptyset$. Then there is an element b of B such that

$$\{y\in Y;\ y\leqq b\}\subset B\setminus f(a_0).\square$$

Proof. By Lemma 5, there is some a_1 in X such that $a_1\geqq a_0$ and a_1 is a maximal element in the sense of Zorn in the set $\{x\in X;\ B\setminus f(x)\neq\emptyset\}$. Then $B\setminus f(a_1)\neq\emptyset$. Let $b\in B\setminus f(a_1)$, and suppose $b'\in f(a_0)\cap\{y\in Y;\ y\leqq b\}$. Then $b'\in f(a_0)\subset f(a_1)$ and $b'\leqq b$. By

Definition 3, there is a subset S of Y such that $b'\in S$ and $b\in(S)$. By the same definition, $b\in(S)$ implies $S\subset\{y\in Y;\ y\leqq b\}$. Since $b\in B$, we conclude that $S\subset B$. On the other hand, the inclusion $S\subset f(a_1)$ is impossible as this inclusion and $b\in(S)$ would imply $b\in f(a_1)$, which is not true. Since $b'\in f(a_1)\cap S$, we have $f(a_1)\cap S\neq\emptyset$ and the previous conclusion gives $S\setminus f(a_1)\neq\emptyset$. Hence, there is some a_2 in X such that $a_2\geqq a_1$, $b\in f(a_2)$ and $S\setminus f(a_2)\neq\emptyset$. The last implies $B\setminus f(a_2)\neq\emptyset$. Using the maximality of a_1, we get $a_2\leqq a_1$. Then $f(a_2)\subset f(a_1)$, which is impossible since $b\in f(a_2)$ and $b\notin f(a_1)$. Thus we proved that there is no b' in $f(a_0)\cap\{y\in Y;\ y\leqq b\}$, i. e.

$$\{y\in Y;\ \ y\leqq b\}\subset B\setminus f(a_0).\square$$

We are now ready for the proof of our main Theorem.

Proof of The Principle of Topological Induction. By Lemma 4, the set Ex of the minimal elements of Y in the sense of Zorn is not empty.

For proving Statement 1 of the Theorem, suppose $p\in\text{Ex}$. Let $B=\{y\in Y;\ y\leqq p\}$. Then B is closed by Lemma 3 and $\{y\in Y;\ y\leqq b\}=B$ for all b in B. Using the last equality and Lemma 6, we conclude that $f(a_0)\cap B=\emptyset$, whenever $a_0\in X$ and $B\setminus f(a_0)\neq\emptyset$. Therefore, $a_0\in X$ and $f(a_0)\cap B\neq\emptyset$ always imply $B\subset f(a_0)$. Hence $y_1\approx y_2$ for all y_1 and y_2 in B. In particular, $y\approx p$ for all y in B.

For the proof of Statement 2 of the Theorem, suppose $a_0\in X$, $\text{Ex}\subset f(a_0)$ but $Y\setminus f(a_0)\neq\emptyset$. Then, applying Lemma 6 for $B=Y$, we conclude that $f(a_0)\cap\{y\in Y;\ y\leqq b\}=\emptyset$ for some b in Y and hence we have $\text{Ex}\cap\{y\in Y;\ y\leqq b\}=\emptyset$ for the same b. This contradicts Lemma 4.

The proof of the Theorem is completed.□

Let us note that Statement 1 of the Theorem shows that all elements of Ex are irreducible in the sense of [2]. Consequently, the main result of [2] may be considered as a corollary of this Theorem.

Finally, we shall indicate how the theorem of Krein-Milman may be obtained from the Principle of Topological Induction.

Let M be a convex compact set in a linear topological space E which is locally convex.

Following the notations of the Theorem we set $X=\{U;\ U$ is open convex set in $E\}$. The order in X is the natural order of inclusion of sets in E.

Let $Y=M$. For each U in X we set $f(U)=U\cap M$.

Now the Theorem implies that if an open convex set U contains the set Ex of minimal points of $Y=M$, then U contains M.

It is not difficult to see that

$$(\{a,b\})=\{ta+(1-t)b;\ 0<t<1\};\ \ a,b\in M,$$

where $(\{a,b\})$ should be understood in the sense of Definition 2.

Since every two points in M may be separated through open convex set in E, the above equality implies that the set Ex is contained in the set of extreme points of M in the usual sense. This immediately gives the Theorem of Krein-Milman: The convex closed hull of the extreme points of M coincides with M.

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